# ROTATION OF THE PLANE OF A CIRCULAR SATELLITE ORBIT

**(POVOROT PLOSKOSTI KRUGOVOI ORBITY SPUTNIKA)**

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**1.** Let us study the motion of an orbiting vehicle in a circular orbit subject to a lateral force $F$. The vector of this force lies at all times in the plane of the horizon and is directed normal to the velocity vector. For the chosen direction of the force it can be easily shown that the velocity modulus and the orbit altitude remain constant during the motion process and equal to the initial values

$$r = r_0 = \text{const}, \qquad V_0 = \sqrt{g_0 r_0} = \text{const}$$

In the general case of a point mass in a central gravitational field the system of equations is [1]

$$\ddot{r} - r\dot{\varphi}^2 \cos^2\theta - r\dot{\theta}^2 = -\frac{\mu}{r^2} + \frac{f_r}{m}$$

$$2\dot{r}\dot{\varphi}\cos\theta + r\ddot{\varphi}\cos\theta - 2r\dot{\varphi}\dot{\theta}\sin\theta = \frac{f_\varphi}{m}$$

$$2\dot{r}\dot{\theta} + r\dot{\varphi}^2 \sin\theta\cos\theta + r\ddot{\theta} = \frac{f_\theta}{m} \qquad (1.1)$$

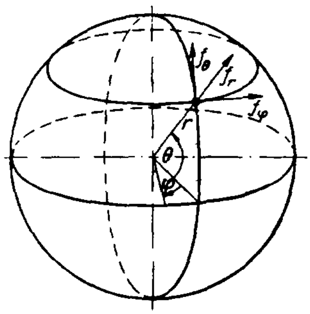


**Fig. 1.**

Here $\mu$ is the gravitational constant, $r$, $\phi$ and $\theta$ are spherical coordinates of a point, where $\theta$ is the latitude (Fig. 1), measured from the equatorial plane; $f_r$, $f_\phi$, $f_\theta$ are the coordinate components of the external force; $m$ is the mass of the vehicle.

Without limiting the generality, we shall assume that the unperturbed circular orbit lies in the equatorial plane.

The initial conditions of system (1.1) are

$$r = r_0, \quad \dot{r} = 0, \quad \varphi = 0, \quad \dot{\varphi} = \omega_0, \quad \theta = 0, \quad \dot{\theta} = 0 \quad \textbf{at} \quad t = 0 \qquad (1.2)$$

$$(\omega_0 = \sqrt{g_0/r_0})$$

Here $\omega_0$ is the angular velocity of the vehicle along a circular orbit of radius $r_0$. The coordinate components of the lateral force (Fig. 2) can be expressed as follows

$$f_r = 0, \qquad f_\varphi = -F\omega_0^{-1}\,\dot{\theta}, \qquad f_\theta = F\,\omega_0^{-1}\,\dot{\varphi}\cos\theta \qquad (1.3)$$

Taking into account (1.3) with the system (1.1), one can derive an equation on $\theta$, which determines the deflection of the vehicle from the initial orbit:

$$\ddot{\theta} - \dot{\theta}^2 \tan\theta + \omega_0^2 \tan\theta = \omega_0 n_z \sqrt{\omega_0^2 - \dot{\theta}^2}, \qquad n_z = F/mg_0 \qquad (1.4)$$

where $n_z$ is the lateral thrust acceleration.

Let us assume that the lateral thrust acceleration $n_z$ is constant during the motion process. Let us introduce into the analysis the function $u(\theta) = \sqrt{(\omega_0^2 - \dot{\theta}^2)}$. Then Equation (1.4) can be reduced to a linear form and its solution becomes:

$$u = \omega_0 \frac{1 - n_z \sin\theta}{\cos\theta} \qquad (1.5)$$

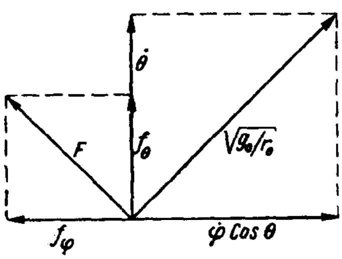


Fig. 2.

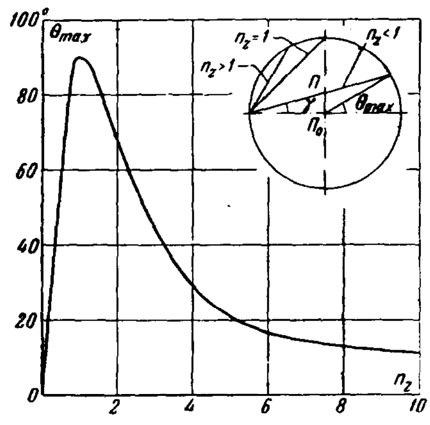


Fig. 3.

After a reversal to the variable $\theta = \theta(t)$, taking into account the initial conditions (1.2), the solution of Equation (1.4) will be

$$\sin\theta = \frac{n_z}{1 + n_z^2}\left(1 - \cos\sqrt{1 + n_z^2}\,\omega_0 t\right) \qquad (1.6)$$

For $n_z \ll 1$ the formula obtained by L.A. Simonov is valid

$$\sin\theta = n_z(1 - \cos\omega_0 t) \tag{1.7}$$

It follows from (1.6) that the angle $\theta$ is a periodic function of time (Fig. 3), whose period $\tau$ and amplitude $\theta_{max}$ are, respectively

$$\tau = \frac{2\pi}{\omega_0\sqrt{1+n_z^2}}, \qquad \theta_{max} = \sin^{-1}\frac{2n_z}{1+n_z^2}$$

Using Expressions (1.5) and (1.6), one can obtain the solution for the angle $\phi$.

Thus, the general solution of the system (1.1), which determines the motion of the vehicle under the action of a lateral force and which lies initially on a circular equatorial orbit, is as follows:

$$r = r_0$$

$$\theta = \sin^{-1}\frac{n_z}{1+n_z^2}\left(1 - \cos\sqrt{1+n_z^2}\,\omega_0 t\right) \tag{1.8}$$

$$\varphi = \tan^{-1}\frac{\sqrt{1+n_z^2}\sin\sqrt{1+n_z^2}\,\omega_0 t}{n_z^2 + \cos\sqrt{1+n_z^2}\,\omega_0 t}$$

**2.** A study of the solution obtained above shows that the torsion $T$ of the trajectory (1.8), which describes its departure from a plane curve, is equal to zero. Indeed,

$$T = \frac{\rho^2}{(x'^2+y'^2+z'^2)^3}\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} \tag{2.1}$$

Here $\rho$ is the radius of curvature of the curve and $x$, $y$, $z$ are the coordinates of the vehicle in a Cartesian coordinate system

$$x = r\cos\theta\cos\varphi = r\left(1 - \frac{\sin\theta}{n_z}\right)$$

$$y = r\cos\theta\sin\varphi = \frac{r}{n_z}\sqrt{\cos^2\theta - (1 - n_z\sin\theta)^2} \tag{2.2}$$

$$z = r\sin\theta.$$

It can be easily seen that the determinant in (2.1) is equal to zero, since when the determinant is expanded in terms of the elements of the second column, all its minors are equal to zero. Consequently, $T = 0$, i.e. the trajectory of the flight of the vehicle is a plane curve.

Applying the usual tools of analytical geometry we can find the equation of the plane $\Pi$ (Fig. 3) which contains the trajectory given by (1.8). We obtain

$$n_z x + z - n_z r_0 = 0 \tag{2.3}$$

The plane (2.3) is inclined to the plane of the unperturbed orbit $\Pi_0 (z = 0)$ at an angle

$$\gamma = \tan^{-1} n_z \tag{2.4}$$

where

$$\gamma = \tfrac{1}{2}\theta \max \quad \text{at } n_z \leqslant 1$$

$$\gamma = \tfrac{1}{2}\pi - \tfrac{1}{2}\theta \max \quad \text{at } n_z \geqslant 1$$

The radius of curvature of the trajectory during motion remains, of course, constant and equal to

$$\rho = \frac{r_0}{\sqrt{1 + n_z^2}} \tag{2.5}$$

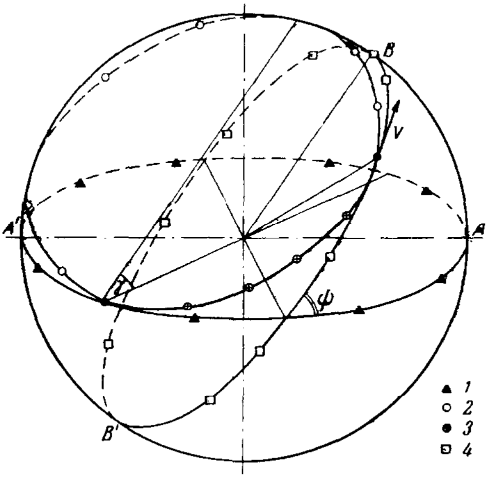

Fig. 4.

Thus, under the action of a constant lateral thrust acceleration $n_z$, the orbital plane of the vehicle which initially moves along the circular orbit $\Pi_0$ of radius $r_0$ (Fig. 3), will rotate by an angle $\gamma$, given by (2.4), and the vehicle will continue its motion in the plane of the small circle with an orbital radius $\rho = r_0 \cos \gamma$. The velocity modulus and the altitude of vehicle flight remain constant and equal to the respective initial values. The perturbed vehicle trajectory is tangent to the original orbit at the point $t = 0$, i.e. after every revolution the vehicle returns to the initial point of the unperturbed orbit.

**3.** The foregoing discussion referred to the case of a continuous action of a lateral force. After a discontinuation of the action of this force, the vehicle orbit will again become central and inclined relative to the unperturbed orbit by some angle $\psi$, as shown in Fig. 4. There *1* is the original orbit, *2* is the orbit with the presence of a constant lateral thrust, *3* is a section of powered flight, *4* is the orbit after a discontinuation of the thrust, and $AB$ and $A'B'$ are the maximum elevations of the vehicle.

Let us study the problem of determining $\psi$ under the condition that the lateral acceleration is produced by a direct thrust of an engine with a given propellant mass fraction $\alpha$ = const. Then the time of operation of the engine $t$ and the constant lateral thrust acceleration $n_z$ produced by it will be related by

$$t_a = -\frac{J}{n_z}\ln(1-\alpha) \tag{3.1}$$

Here $J$ is the specific impulse of the engine. The vehicle trajectory consists of two parts in the case of a limited time of thrust action; the active part *3* and the passive part *4* (Fig. 4). Of course, the magnitude of the angle $\psi$ will be equal to the maximum value of the elevation angle of the vehicle from the original orbit, $\theta_{max}$, along the passive segment of the trajectory at the instant where $\dot{\theta} = 0$. This value of $\psi = \theta_{max}$ can be found from Equation (1.1) with $f_r = 0$, $f_\phi = 0$, and $f_\theta = 0$ and the initial conditions, which correspond to the values of the functions $r$, $\theta$ and $\phi$ and their derivatives given by (1.8) at the end of the active segment at $t = t_a$.

As was shown above, the velocity of motion along the active segment *3* is equal to the circular velocity corresponding to the flight altitude of the vehicle, where the altitude and the velocity modulus remain constant during the motion. Consequently, after the discontinuation of the lateral thrust, the vehicle will move along a circular orbit *4* with the previous values of velocity modulus and altitude.

When the above is considered one can derive the following equation from (1.1):

$$\ddot{\theta} - \dot{\theta}^2 \tan\theta + \omega_0^2 \tan\theta = 0 \tag{3.2}$$

After the substitution $p(\theta) = \dot{\theta}\cos\theta$, Equation (3.2) reduces to a first order equation with separable variables. After some simple transformations we obtain the first integral of Equation (3.2)

$$\dot{\theta} = \frac{1}{\cos\theta}\left(\dot{\theta}_a^2\cos^2\theta_a - \frac{1}{2}\omega_0^2\cos 2\theta_a + \frac{1}{2}\omega_0^2\cos 2\theta\right)^{\frac{1}{2}} \tag{3.3}$$

where $\theta_a$ and $\dot{\theta}_a$ are the values of the elevation angle and its derivative at the end of the active segment of the trajectory when $t = t_a$.

Equating (3.3) to zero, we obtain a relation for the determination of the quantity $\psi = \theta_{max}$

$$\cos\psi = 1 - n_z\sin\theta_a \tag{3.4}$$

or, considering (1.8), we obtain

$$\sin\frac{\psi}{2}=\frac{n_z}{\sqrt{1+n_z^2}}\sin\frac{1}{2}\sqrt{1+n_z^2}\,\omega_0 t_a \tag{3.5}$$

The available propellant mass fraction can be characterized by the ideal velocity $v$, which the vehicle acquires after expelling the propellant supply in vacuum at a specific impulse $J$

$$\alpha=1-\exp\left(-\frac{v}{Jg_0}\right) \tag{3.6}$$

Then the maximum deflection of the vehicle from the unperturbed orbit with a given $v$ will be from (1.8), (3.1), (3.5) and (3.6)

$$\sin\frac{\psi}{2}=\frac{n_z}{\sqrt{1+n_z^2}}\sin\frac{\sqrt{1+n_z^2}}{2n_z}\frac{v}{V_0} \tag{3.7}$$

An analysis of Equation (3.7) (curve *1*, Fig. 5) shows that with small values of $n_z$, the maximum angle of rotation $\psi$ of the orbital plane of the vehicle from the original orbit is a periodic function that damps out as $n_z \to 0$. This is explained by the fact that when a constant direction of the lateral thrust during active flight is maintained, as explained above, and the vehicle moves along a small circle of the terrestrial sphere, it will return periodically once every revolution to the original point, while the reduction of the angle $\psi$ takes place during the even half periods. Thus, in order to increase the total deflection angle of the orbital plane of the vehicle it is expedient to reverse the direction of thrust during each half-period.

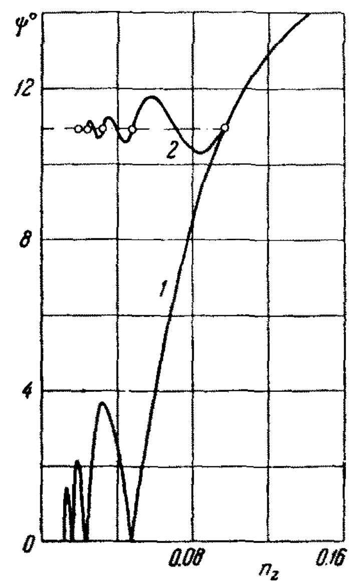

Fig. 5.

Then, after an integral number of half-periods, $N$, the rotation angle will be equal to

$$\psi=N\sin^{-1}\frac{2n_z}{1+n_z^2} \tag{3.8}$$

where, of course, $\phi=N\pi$. The general time of flight of the vehicle is determined by the Expression (3.1), and the time of one half-revolution from (1.6)

$$\frac{\tau}{2}=\frac{\pi}{\omega_0\sqrt{1+n_z^2}} \tag{3.9}$$

If $t_a$ is a multiple of $\tau/2$, then, determining the number of possible half-revolutions $N$ and substituting its value into (3.8), we obtain

$$\psi = \frac{1}{\pi}\frac{v}{V_0}\frac{\sqrt{1+n_z^2}}{n_z} \sin^{-1} \frac{2n_z}{1+n_z^2} \tag{3.10}$$

For small values of $n_z$ we have

$$\psi \approx \frac{2}{\pi}\frac{v}{V_0} \tag{3.11}$$

An analysis of (3.10) shows that with thrust acceleration smaller than $n_z$ corresponding to the expulsion of the entire fuel supply during the first half-revolution $n_z(\tau/2)$, and with an assumed condition of a short general time of the powered flight in one half-period, the magnitude of possible orbital plane rotation of the vehicle stays practically constant (points on Fig. 5).

The magnitude of the lateral force, which corresponds to the expulsion of the propellant during the first half-revolution, follows from (3.1), (3.6) and (3.9)

$$n_z(\tau/2) = \left[\left(\frac{\pi V_0}{v}\right)^2 - 1\right]^{-\frac{1}{2}} \tag{3.12}$$

With an increase of $n_z$, the value of the angle $\psi$ grows and approaches asymptotically some limiting value. Physically this means that the faster the available propellant is being expelled (i.e. the larger the thrust acceleration $n_z$) the more will the vehicle deflect from the unperturbed orbit. The maximum deflection of the vehicle with a fixed propellant supply takes place when $n_z \to \infty$, i.e. in the case of an impulsive application of thrust

$$\psi \doteq \frac{v}{V_0} \quad \text{as } n_z \to \infty \tag{3.13}$$

A comparison of (3.11) and (3.13) shows that with a given characteristic velocity $v$ in the case of an impulsive action, the angle of orbit rotation is $\pi/2$ times larger than that with the studied small accelerations.

**4.** In general, the condition that the time of active flight $t_a$ is not an integral multiple of the time of a half-period of the perturbed orbit is not satisfied. In order to determine the maximum possible orbital plane rotation in this case one can pursue the following course. At first we calculate the rotation angle of the orbital plane for an integral number of active half-revolutions $\theta^*$, given by equation (3.8). During the flight along this part of the trajectory a reversal of the direction of the lateral force in each half-revolution takes place at $\phi = N\pi$.

We have the following initial conditions for the last portion of the active segment

$$t = N\tau/2, \qquad \theta = \theta^*, \qquad \dot{\theta} = 0, \qquad \varphi = N\pi \tag{4.1}$$

The time of active flight in this segment will be, of course

$$\Delta t = t_a - N\tau/2 < \tau/2 \tag{4.2}$$

Thus, we arrive at a problem that is similar to the one studied above, but with different initial conditions on $\theta$ and $\phi$. The rotation angle of the orbital plane $\psi$ will be equal, as before, to the maximum elevation angle of the vehicle from the initial orbit $\theta_{max}$ which takes place along the passive part of the trajectory at the instant corresponding to $\dot{\theta} = 0$. When we carry out calculations that are similar to the previous ones, with the exception of new initial conditions, we obtain expressions for the determination of the parameters at the end of the active segment

$$\dot{\theta}_a = \omega_0 \left[ 1 - \left( \frac{\cos\theta^* + n_z \sin\theta^*}{\cos\theta_a} - n_z \tan\theta_a \right)^2 \right]^{1/2} \tag{4.3}$$

$$\theta_a = \sin^{-1} \frac{n_z(\cos\theta^* + n_z \sin\theta^*) + (\sin\theta^* - n_z\cos\theta^*)\cos\vartheta}{1 + n_z^2} \tag{4.4}$$

$$\varphi_a = \tan^{-1} \frac{\sqrt{1+n_z^2}\sin\vartheta}{(\cos\theta^* + n_z\sin\theta^*)\cos\vartheta - n_z(\sin\theta^* - n_z\cos\theta^*)} \tag{4.5}$$

Here

$$\vartheta = \sqrt{1+n_z^2}\,\omega_0 \Delta t$$

Furthermore, studying the unpowered flight with the initial conditions, given by Expressions (4.3) to (4.5), we obtain in parametric form expressions for the vehicle trajectory after the termination of lateral thrust

$$\theta = \sin^{-1} [\dot{\theta}_a \omega_0^{-1} \cos\theta_a \sin\omega_0 (t - t_a) + \sin\theta_a \cos\omega_0 (t - t_a)] \tag{4.6}$$

$$\varphi = \varphi_a + \tan^{-1} \left[ \cos\theta_a \frac{\omega_0 \dot{\theta}_a^{-1} \tan\theta_a + \tan\omega_0 (t - t_a)}{1 - \omega_0 \dot{\theta}_a^{-1} \tan\theta_a \tan\omega_0 (t - t_a)} \left( 1 - \frac{\dot{\theta}_a^2}{\omega_0^2} \right)^{1/2} \right] - $$
$$- \tan^{-1} \left( 1 + \frac{\dot{\theta}_a^2}{\omega_0^2} \right)^{-1/2} \sin\theta_a \tag{4.7}$$

Equating $\dot{\theta}$ to zero (see (3.3)) and using (4.3), we obtain an expression for the determination of the unknown rotation angle $\psi$ of the orbital plane of the satellite

$$\cos 2\psi = 2\,[\cos\theta^* + n_z(\sin\theta^* - \sin\theta_a)]^2 - 1 \qquad (4.8)$$

An analysis of Equation (4.8) (curve 2, Fig. 5) shows that for values of $n_z$ smaller than $n_z(r/2)$, given by Formula (3.12), the values of the rotation angle of the orbital plane $\psi$ oscillate around the values given by Formula (3.10), which gives $\psi$ for a value of $n_z$ corresponding to the expulsion of the propellant supply in an integral number of half-periods. The amplitude of the oscillations of $\psi$ in the case shown in Fig. 5 for $v/V_0 = 0.3$ does not exceed 1 degree and decreases with a decrease of $n_z$. The oscillation period is equal to the difference between the successive working times of the engine corresponding to the expulsion of the entire propellant supply in an integral number of half-revolutions. Thus, just as the amplitude, the period of the oscillations also decreases with a decrease in the thrust acceleration $n_z$.

**5.** In conclusion, let us note that when utilizing aerodynamic forces for the creation of the lateral force a rotation angle of the orbital plane with a given supply of propellant will vary in direct proportion to the magnitude of the aerodynamic effect, since the use of the aerodynamic effect with a given lateral acceleration can be treated as a corresponding change of the available characteristic velocity.

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